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LETTER TO THE EDITOR

Inhomogeneous solutions of the Boltzmann equation with external forces in a time-dependent host medium

M L Martiarena† and R O Barrachina‡
Centro Atómico Bariloche‡, 8400 Bariloche, Argentina

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Abstract. A gas of test particles in d dimensions, which interacts with a background host medium in the presence of external forces, is studied. Scattering, removal and self-generation collisions are taken into account. A generalised Nikolskii transform reduces the inhomogeneous Boltzman equation for this system to the homogeneous one without background or external forces. We show that a background of field particles can confine the test gas, even in the absence of external forces.

The non-linear Boltzmann equation (NLBE) describes the temporal evolution of a gas of particles interacting through binary collision. In recent years, the discovery of an exact particular solution of the spatially uniform NLBE for $1/|\boldsymbol{v}|$ cross sections (Bobylev 1976, Krook and Wu 1976) initiated a revival of interest in this non-linear relaxation problem. This research has been mainly limited to a spatially uniform gas with elastic collision (Ernst 1981). However, it would be desirable to describe more realistic physical situations, namely an inhomogeneous gas interacting with external forces or with a background host medium (Spiga *et al* 1985, Menon 1985). Actually it is of real interest to study the asymptotic behaviour of physical systems subjected to such external perturbations (Haken 1983).

The Nikolskii transform makes it possible to build up a class of inhomogeneous solutions of the NLBE starting from homogeneous ones (Nikolskii 1964). Unfortunately these solutions describe a gas in expansion, with vanishing density and temperature. Recently Cornille (1985, 1986) showed that these solutions can be confined by external forces, in the sense that the gas approaches an absolute Maxwellian distribution.

In this letter we show that it is possible to obtain a confined inhomogeneous gas, in the sense described above, when the gas interacts with a background host medium. This behaviour is possible even if there are no external forces applied to the system.

Furthermore we show that a generalised Nikolskii transform reduces the inhomogeneous Boltzmann equation in the presence of a background host medium and external forces to the homogeneous one without such perturbations. This is an important result since this latter equation has been extensively studied and several solutions are known (Ernst 1981). Theorems of existence and uniqueness as well as numerical and approximate solutions have been proposed (Fujii *et al* 1986, Barrachina and Garibotti 1986).

† Consejo Nacional de Investigacionnes Cientificas y Técnicas, Argentina.
‡ Comisión Nacional de Energia Atómica, Argentina.

We study a gas of test particles (TP) in d dimensions, which interacts with a background of field particles (FP) in the presence of an external force, including scattering, removal (e.g. chemical reactions) and self-generation collision of the TP among themselves as well as with the FP of the host medium. In this context self-generation means that TP can be produced by collisions between themselves (e.g. phonon or photon gases, electroweak processes, biological systems (Schuster 1986)) or against the background (e.g. neutron-induced fission or electron-induced ionisation).

The NLBE that governs the temporal evolution of the test particle distribution function is

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\boldsymbol{r}, \boldsymbol{v}, t) = B[f, f] + L[f, t] - f(\boldsymbol{r}, \boldsymbol{v}, t) \int \mu_{\mathrm{R}}(|\boldsymbol{v} - \boldsymbol{w}|) f(\boldsymbol{r}, \boldsymbol{w}, t)\, \mathrm{d}\boldsymbol{w}$$
$$- f(\boldsymbol{r}, \boldsymbol{v}, t) \int \nu_{\mathrm{R}}(|\boldsymbol{v} - \boldsymbol{w}|) g(\boldsymbol{r}, \boldsymbol{w}, t)\, \mathrm{d}\boldsymbol{w} \tag{1}$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla_r + \boldsymbol{a}(r, t) \cdot \nabla_v \tag{2}$$

the free-streaming operator, $g(\boldsymbol{r}, \boldsymbol{v}, t)$ the distribution function of the background gas and $\boldsymbol{a}(\boldsymbol{r}, t)$ a general external force per mass unit. The factors $\nu_{\mathrm{R}}(v)$ and $\mu_{\mathrm{R}}(v)$ represent the removal and regeneration collision frequencies of TP with FP and among themselves, respectively. $B[f, f]$ is the bilinear term associated with the TP–TP scattering collisions:

$$B[f, f] = \int \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\hat{n} |\boldsymbol{v} - \boldsymbol{v}_1| \sigma\left(|\boldsymbol{v} - \boldsymbol{v}_1|, \frac{(\boldsymbol{v} - \boldsymbol{v}_1) \cdot \hat{n}}{|\boldsymbol{v} - \boldsymbol{v}_1|} \right)$$
$$\times [f(\boldsymbol{r}, \boldsymbol{v}', t) f(\boldsymbol{r}, \boldsymbol{v}_1', t) - f(\boldsymbol{r}, \boldsymbol{v}, t) f(\boldsymbol{r}, \boldsymbol{v}_1, t)] \tag{3}$$

where $\sigma\ (v, \cos\theta)$ is the cross section.

The incoming and post-collisional velocities are related by the dynamics:

$$\boldsymbol{v}' = \tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{v}_1) + \tfrac{1}{2}|\boldsymbol{v} - \boldsymbol{v}_1| \hat{n} \tag{4a}$$
$$\boldsymbol{v}_1' = \tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{v}_1) - \tfrac{1}{2}|\boldsymbol{v} - \boldsymbol{v}_1| \hat{n} \tag{4b}$$

with $\hat{n}$ a unit vector in the direction of the post-collisional relative velocity $\boldsymbol{v}' - \boldsymbol{v}_1'$.

$L[f, t]$ is the linear collision operator which describes the TP–FP elastic interactions. If we suppose that the ratio of the FP mass and TP mass approaches zero, then $L[f, t] = 0$ (Menon 1985). This is the case we will consider in the present letter.

Furthermore, we suppose that the removal and regeneration frequencies $\nu_{\mathrm{R}}(v)$ and $\mu_{\mathrm{R}}(v)$ are constants (Maxwell models).

Then (1) reduces to

$$[\mathrm{D}/\mathrm{D}t + \mu_{\mathrm{R}}\rho + \nu_{\mathrm{R}} n] f(\boldsymbol{r}, \boldsymbol{v}, t) = B[f, f] \tag{5}$$

with $n(\boldsymbol{r}, t) = \int g(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}$ and $\rho(\boldsymbol{r}, t) = \int f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}$ the density of field particles and test particles, respectively. For simplicity we suppose that the FP density $n(t)$ is a function of time only.

Now we look for an inhomogeneous similarity solution of the NLBE (5) such that the space variable appears in the distribution function only through the bulk velocity:

$$\boldsymbol{u}(\boldsymbol{r}, t) = \frac{1}{\rho(\boldsymbol{r}, t)} \int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}. \tag{6}$$

Then we write

$$f \equiv f(c(\boldsymbol{r}, \boldsymbol{v}, t), t) \tag{7}$$

with $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r}, t)$ the peculiar velocity. This ansatz is suggested by a local Maxwellian description of the gas (Cornille 1986). It is easy to see that, for this similarity solution, the moments formed with the peculiar velocity may be a function of time but not of space coordinates; namely, the density ρ, temperature T, stress tensor $\vec{P}$ and make the continuity equation compatible div $\boldsymbol{u}$ has to be space independent. Thus we consider a flow for which the bulk velocity is a linear function of position:

$$\boldsymbol{u}(\boldsymbol{r}, t) = \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\boldsymbol{r} + \boldsymbol{u}_0(t) \tag{8}$$

with $\gamma(0) = 0$. The balance of momentum immediately shows that the most general external force, compatible with (8), is a pure harmonic one:

$$\boldsymbol{a}(\boldsymbol{r}, t) = k(t)\boldsymbol{r} + \boldsymbol{a}_0(t). \tag{9}$$

The force alone determines the bulk velocity (8) with

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} = k\gamma \tag{10}$$

$$\frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}t} + \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\boldsymbol{u}_0 = \boldsymbol{a}_0. \tag{11}$$

For certain functional forms of the elastic constant $k(t)$ this equation can be solved using appropriate special functions. This solution exists for such t that keeps (t) positive.

In continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \left(\nu_{\mathrm{R}} n + \frac{d}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)\rho + \mu_{\mathrm{R}}\rho^2 = 0 \tag{12}$$

is a non-linear ordinary first-order differential equation of Ricatti type. We remark that this kind of equation is typical in the context of dynamical systems (Haken 1983). Its solution is (Zanette and Barrachina 1987)

$$\rho(t) = \frac{\rho(0)\gamma(t)^{-d}\exp(-\nu_{\mathrm{R}}\int_0^t n(t')\,\mathrm{d}t')}{1 + \mu_{\mathrm{R}}\rho(0)\int_0^t \gamma(t)^{-d}\exp(-\nu_{\mathrm{R}}\int_0^{t'} n(t'')\,\mathrm{d}t')}. \tag{13}$$

Finally the conservation of energy immediately leads to the following expression for the temperature:

$$T(t) = T(0)\gamma(t)^{-2}. \tag{14}$$

Let us restrict the similarity solution (7) further by means of a generalised Nilkoskii transformation:

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = \rho(t)\gamma(t)^d F(\boldsymbol{\eta}, t) \tag{15}$$

with

$$\boldsymbol{\eta}(\boldsymbol{r}, \boldsymbol{v}, t) = \gamma(t)(\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r}, t)).$$

We rewrite the NLBE for this inhomogeneous similarity solution:

$$\partial F/\partial t = \rho(t)\gamma^{2(d-1)/(p-1)-1}B[F, F] \tag{16}$$

where we have assumed intermolecular forces with an inverse power law p.

Finally, by redefining the timescale

$$\tau = \int_0^t \rho(t)\gamma^{2(d-1)/(p-1)-1}\,\mathrm{d}t \tag{17}$$

we obtain

$$\partial F/\partial\tau = B[F, F]. \tag{18}$$

We have reduced the study of the generalised inhomogeneous NLBE (5) to the homogeneous NLBE without creation or destruction of particles and without external forces. This is a very important result since (18) has been extensively studied and several solutions have been found.

Calculating the first moments of $\boldsymbol{\eta}$, we obtain the usual conservation laws of particles, momentum and energy for a homogeneous gas:

$$\int F(\boldsymbol{\eta}, t)\,\mathrm{d}\boldsymbol{\eta} = 1 \tag{19a}$$

$$\int F(\boldsymbol{\eta}, t)\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{\eta} = 0 \tag{19b}$$

$$\int F(\boldsymbol{\eta}, t)\eta^2\,\mathrm{d}\boldsymbol{\eta} = T(0)d. \tag{19c}$$

It is important to note that the external force alone determines $\boldsymbol{u}(\boldsymbol{r}, t)$ and $T(t)$. The density $\rho(t)$ is mainly controlled by creation and destruction events instead. It is bounded for μ_{R} positive and its large time evolution can sharply change depending on the limiting behaviour of

$$\nu(t) = \nu_{\mathrm{R}} n(t) + \frac{d}{\gamma(t)}\frac{\mathrm{d}\gamma}{\mathrm{d}t}. \tag{20}$$

Actually, a non-vanishing density can be obtained when TP–FP regeneration collisions are allowed. In figure 1, we display such a behaviour for the choice $\nu_{\mathrm{R}} < 0$, $n(t) = cte$

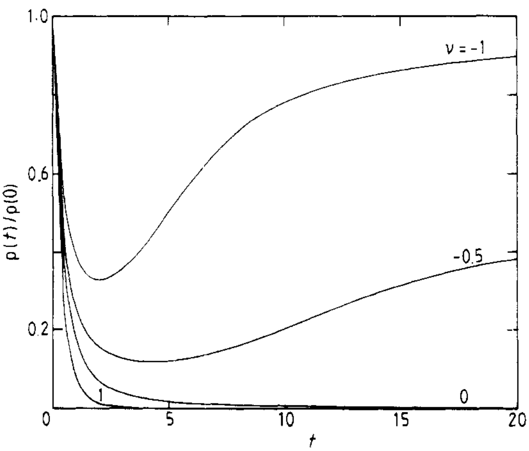

Figure 1. Evolution of the test particles density when self-removal of TP is allowed ($\mu_{\mathrm{R}} > 0$). We consider dimensionless variables, $t/\mu_{\mathrm{R}}\rho(0)$ and $\mu_{\mathrm{R}}\rho(0)\nu$. The field particle density is $\nu(t) = cte$.

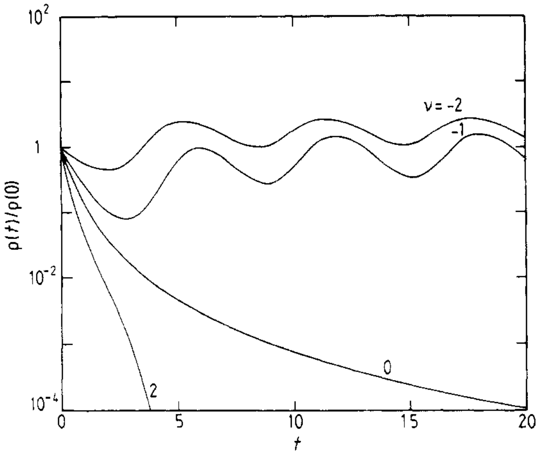

Figure 2. As in figure 1, with an oscillating field particle density $\nu(t) = n_0 + \sin(t)$.

in the absence of external forces. Another important situation arises in the study of an oscillatory FP density $n = n_0 + n_1 \sin(\omega t)$. We observe in figure 2 that, after a transient, the TP density reaches a limit cycle around the equilibrium point $\rho_0 = |\nu_R| n_0 / \mu_R$.

When self-generation collisions are allowed, i.e. $\mu_R < 0$, the quadratic term in (12) contributes to increasing the density. In fact, if $\nu(t)$ satisfies

$$|\mu_R|\rho(0) \int_0^\infty \exp\left(-\int_0^t \nu(t')\,dt'\right) dt > 1 \tag{21}$$

a divergence $\rho(t) = (t - t_0)^{-1}$ will take place in a finite time.

In the absence of external forces

$$\gamma(t) = 1 + \left.\frac{d\gamma}{dt}\right|_0 t \tag{22}$$

and the temperature of the test gas goes to zero when $t \to \infty$. However, when TP-FP creation collisions occur, the background host medium can confine the test gas, even in the absence of external forces. Actually, the gas freezes in a Dirac-type equilibrium distribution function:

$$f(r, v, t) \xrightarrow[t\to\infty]{} \rho(t)\delta(\boldsymbol{v}). \tag{23}$$

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